

# Normal states of type III factors

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## Abstract

Let  $M$  be a factor of type III with separable predual and with normal states  $\varphi_1, \dots, \varphi_k, \omega$  with  $\omega$  faithful. Let  $A$  be a finite dimensional  $C^*$ -subalgebra of  $M$ . Then it is shown that there is a unitary operator  $u \in M$  such that  $\varphi_i \circ \text{Ad } u = \omega$  on  $A$  for  $i = 1, \dots, k$ . We also have a similar result for a factor of type  $\text{II}_1$ .

## 1 Introduction

Let  $M$  be a factor of type III with separable predual. Then two nonzero projections  $e$  and  $f$  in  $M$  are equivalent, i.e., there exists a partial isometry  $v \in M$  such that  $v^*v = e$ ,  $vv^* = f$ . If furthermore  $e$  and  $f$  are different from the identity operator 1, then there is a unitary operator  $u \in M$  such that  $u^*eu = f$ .

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This shows that there is an abundance of unitaries in  $M$ , so one might expect stronger results arising from these unitaries. That is what is done in the present paper. We show that if  $\varphi$  and  $\omega$  are faithful normal states in  $M$  and  $A \subset M$  is a finite dimensional  $C^*$ -algebra, then there exists a unitary operator  $u \in M$  such that the restrictions  $\varphi \circ \text{Ad } u|_A$  and  $\omega|_A$  are equal, where  $\text{Ad } u$  is the inner automorphism  $x \mapsto u^*xu$  of  $M$ . (See Corollary 2.2 for a more precise and general statement.) This result is then applied to obtain a similar result for the compact operator on a separable Hilbert space.

If  $M$  is not of type III, the corresponding result is false in general, but if  $M$  is a factor of  $\text{II}_1$  and  $\omega = \tau$  is the trace and  $A \cong M_n(\mathbb{C})$ , the matrix algebra of complex  $n \times n$ -matrices, then the corresponding result holds for  $\omega = \tau$  and any  $\varphi$ . This will be shown in Section 3.

There exist results of a similar nature to the ones above in the literature. In [CS], it has been shown that if  $M$  is of type  $\text{III}_1$  and  $\varepsilon > 0$  then there is a unitary operator  $u \in M$  with

$$\|\varphi \circ \text{Ad } u - \omega\| < \varepsilon.$$

If one takes a pointwise weak limit point of the automorphisms of the form  $\text{Ad } u$  in the above, then one finds a completely positive unital map  $\pi : M \rightarrow M$  with  $\varphi \circ \pi = \omega$ .

In the non-separable case, it has recently been shown by Ando and Haagerup that for some factors of type  $\text{III}_1$  constructed as ultraproducts, all faithful normal states are unitarily equivalent [AH].

In the  $C^*$ -algebra case it has been shown in [KOS] that if  $\varphi$  and  $\omega$  are pure states of a separable  $C^*$ -algebra  $A$  with the same kernel for their GNS-representations, then there is an asymptotically inner automorphism  $\alpha$  of  $A$  such that  $\varphi \circ \alpha = \omega$ .

Our result gives an exact equality of two states, not an approximate one, but only on a finite dimensional  $C^*$ -subalgebra  $A$ .

## 2 Factors of type III

In this section we state and prove our main result and then apply it to its extension to the compact operators on a separable Hilbert space.

**Theorem 2.1** *Let  $M$  be a type III factor with separable predual, and  $\varphi_1, \dots, \varphi_k$  normal states on  $M$ . Let  $A$  be a finite dimensional  $C^*$ -algebra and  $\rho$  a faithful state on  $A$ . Then there exists a unital injective homomorphism  $\pi : A \rightarrow M$  with*

$$\varphi_i \circ \pi = \rho, \quad i = 1, \dots, k.$$

After proving this theorem, we will prove that it implies the following corollary.

**Corollary 2.2** *Let  $M$  be a factor of type III with separable predual. Let  $A$  be a finite dimensional  $C^*$ -subalgebra of  $M$ . Let  $\varphi_1, \dots, \varphi_k$  and  $\omega$  be normal states*

on  $M$  and assume that  $\omega$  is faithful. Then there exists a unitary operator  $u \in M$  such that

$$\varphi_i \circ \text{Ad } u|_A = \omega|_A, \quad i = 1, \dots, k.$$

Before starting preliminaries of our proof of Theorem 2.1, we give an outline of our method for the case  $A \cong M_d(\mathbb{C})$ .

After diagonalizing the density matrix of  $\rho$ , what we have to find is a system of matrix units  $\{e_{ij}\}$  in  $M$  for which we have  $\varphi_n(e_{ij}) = \delta_{ij}\lambda_i$  for all  $n = 1, \dots, k$  and  $i, j = 1, \dots, d$ , where  $\lambda_i$ 's are eigenvalues of the density matrix of  $\rho$ . We first choose  $e_{ii}$ 's satisfying this condition. Then we choose  $e_{12}, e_{13}, \dots, e_{1d}$  inductively so that we have various identities saying that the values of certain linear functionals applied to a certain partial isometry are all zero at each induction step. This is done by the repeated use of Lemma 2.5, by which we can choose a partial isometry and replace it at each induction step.

The following proposition is used repeatedly in our analysis.

**Proposition 2.3** *Let  $M$  be a  $\sigma$ -finite diffuse von Neumann algebra,  $\varphi_1, \dots, \varphi_k$  be positive normal linear functionals on  $M$ , and  $\psi_1, \dots, \psi_l$  be  $\sigma$ -weakly continuous linear functionals on  $M$  with  $\psi_j(1) = 0$  for all  $j = 1, \dots, l$ . Then there exists a strongly continuous increasing map  $p : [0, 1] \rightarrow \text{Proj } M$  satisfying  $p_0 = 0, p_1 = 1$  and*

$$\begin{aligned} \varphi_i(p_t) &= t\varphi_i(1), \quad i = 1, \dots, k, \\ \psi_j(p_t) &= 0, \quad j = 1, \dots, l. \end{aligned}$$

Before the proof of the Proposition 2.3, we recall the following basic fact.

**Lemma 2.4** *Let  $M$  be a von Neumann algebra and  $\varphi_i, i = 0, \dots, m$  be positive normal linear functionals on  $M$ . Assume that  $\varphi_0$  is faithful. Let  $\{p_{l,j}\}_{j=1, \dots, 2^l}$ ,  $l \in \mathbb{N}$ , be a family of projections in  $M$  such that*

$$p_{l,j} = p_{l+1,2j-1} + p_{l+1,2j}, \quad \sum_{j=1}^{2^l} p_{l,j} = 1,$$

and

$$\varphi_i(p_{l,j}) = \frac{1}{2^l} \varphi_i(1), \quad j = 1, \dots, 2^l, \quad i = 0, \dots, m.$$

Then there exists a strongly continuous increasing map  $p : [0, 1] \rightarrow \text{Proj } M$  satisfying

$$p_0 = 0, \quad p_1 = 1, \quad \varphi_i(p_t) = t\varphi_i(1), \quad \text{for all } t \in [0, 1], \quad i = 0, \dots, m.$$

**Proof.** Note that for a fixed  $t$ , the projections  $\sum_{j \text{ with } t > j/2^l} p_{l,j}$  are increasing in  $l$ . We thus set, for each  $t \in [0, 1]$ ,

$$p_t := \sup_l \sum_{j \text{ with } t > j/2^l} p_{l,j} = \text{s-lim}_l \sum_{j \text{ with } t > j/2^l} p_{l,j} \in \text{Proj } M.$$

Then as  $\varphi_i$ 's are normal, this  $p_t$  satisfies

$$p_0 = 0, \quad p_1 = 1, \quad \varphi_i(p_t) = t\varphi_i(1), \quad \text{for all } t \in [0, 1], \quad i = 0, \dots, m.$$

Furthermore,  $p_t$  is an increasing family of projections by definition. As  $\varphi_0$  is faithful,  $p_t$  is strongly continuous, since the strong operator topology on the unit ball of  $M$  is given by the distance  $\varphi_0((x - y)^*(x - y))^{1/2}$ .  $\square$

Now we prove Proposition 2.3. Recall that the classical theorem of Lyapunov [L], [DU, page 264, Corollary 5] states that for finite atomless measures  $\mu_1, \mu_2, \dots, \mu_n$  on a measure space  $X$ , the set

$$\{(\mu_1(E), \mu_2(E), \dots, \mu_n(E)) \mid E \subset X \text{ is measurable}\}$$

is convex in  $\mathbb{R}^n$ .

**Proof of Proposition 2.3.** Let  $B$  be an abelian diffuse von Neumann subalgebra of  $M$ .

Let  $\eta_i = \operatorname{Re} \psi_i$ ,  $\xi_i = \operatorname{Im} \psi_i$  and  $\eta_i = \eta_{i,+} - \eta_{i,-}$ ,  $\xi_i = \xi_{i,+} - \xi_{i,-}$  be the Jordan decompositions. Note that  $\eta_{i,+}(1) = \eta_{i,-}(1)$ ,  $\xi_{i,+}(1) = \xi_{i,-}(1)$ . Let  $\varphi$  be a faithful normal state on  $M$ . For the restrictions of the positive normal linear functionals

$$\varphi, \varphi_j, \eta_{i,+}, \eta_{i,-}, \xi_{i,+}, \xi_{i,-},$$

on the abelian diffuse subalgebra  $B$ , where  $j = 1, \dots, k$  and  $i = 1, \dots, l$ , the Lyapunov theorem applied to the midpoint of the segment between the points  $(\varphi(1), \varphi_1(1), \dots, \xi_{l,-}(1))$  and  $(\varphi(0), \varphi_1(0), \dots, \xi_{l,-}(0))$ , which is the point  $(0, \dots, 0)$ , we can find a projection  $p_{1,1}$  in  $B$  such that we have

$$\begin{aligned} \varphi(p_{1,1}) &= \frac{1}{2}, \quad \varphi_j(p_{1,1}) = \frac{1}{2}\varphi_j(1), \\ \eta_{i,+}(p_{1,1}) &= \frac{1}{2}\eta_{i,+}(1) = \frac{1}{2}\eta_{i,-}(1) = \eta_{i,-}(p_{1,1}), \\ \xi_{i,+}(p_{1,1}) &= \frac{1}{2}\xi_{i,+}(1) = \frac{1}{2}\xi_{i,-}(1) = \xi_{i,-}(p_{1,1}). \end{aligned}$$

We set  $p_{1,2} = 1 - p_{1,1}$  and apply the same procedure for  $p_{1,1}Mp_{1,1}$  and  $p_{1,2}Mp_{1,2}$  with  $p_{1,1}Bp_{1,1}$  and  $p_{1,2}Bp_{1,2}$ , respectively. Then by induction and Lemma 2.4, we obtain a strongly continuous increasing map  $p : [0, 1] \rightarrow \operatorname{Proj} M$  satisfying  $p_0 = 0, p_1 = 1$  and

$$\begin{aligned} \varphi(p_t) &= t, \quad \varphi_j(p_t) = t\varphi_j(1), \\ \eta_{i,+}(p_t) &= t\eta_{i,+}(1) = t\eta_{i,-}(1) = \eta_{i,-}(p_t), \\ \xi_{i,+}(p_t) &= t\xi_{i,+}(1) = t\xi_{i,-}(1) = \xi_{i,-}(p_t), \end{aligned}$$

for all  $t \in [0, 1]$ ,  $i = 1, \dots, l$  and  $j = 1, \dots, k$ .

From this, we have

$$\psi_i(p_t) = 0, \quad i = 1, \dots, l,$$

and we are done.  $\square$

Now we use the proposition to construct appropriate matrix units. We first note the following fact.

**Lemma 2.5** *Let  $M$  be a  $\sigma$ -finite diffuse von Neumann algebra,  $\psi, \psi_1, \dots, \psi_l$  be  $\sigma$ -weakly continuous linear functionals on  $M$ , and  $v$  a partial isometry in  $M$  satisfying*

$$\psi_i(v) = 0, \quad i = 1, \dots, l.$$

*(Here  $l$  can be 0.) Then there exists a partial isometry  $\bar{v}$  in  $M$  satisfying*

$$\begin{aligned} \psi_i(\bar{v}) &= 0, \quad i = 1, \dots, l, \\ \psi(\bar{v}) &= 0, \\ \bar{v}^* \bar{v} &= v^* v, \\ \bar{v} \bar{v}^* &= v v^*. \end{aligned}$$

**Proof.** We claim there exists a projection  $p$  in  $M$  satisfying  $p \leq v^* v$  and

$$\psi_i(vp) = 0, \quad i = 1, \dots, l,$$

and

$$|\psi(vp)| = |\psi(v(1-p))|.$$

Let  $e := v^* v$ . Then  $\tilde{\psi}_1 := \psi_1(v \cdot)|_{M_e}, \dots, \tilde{\psi}_l := \psi_l(v \cdot)|_{M_e}$  and  $\tilde{\psi} := \psi(v \cdot)|_{M_e}$  are  $\sigma$ -weakly continuous linear functionals on  $M_e$  satisfying

$$\tilde{\psi}_i(e) = \psi_i(v e) = \psi_i(v) = 0, \quad i = 1, \dots, l.$$

By Proposition 2.3, there exists a strongly continuous increasing map  $p : [0, 1] \rightarrow \text{Proj } M_e$  satisfying  $p_0 = 0$ ,  $p_1 = e$  and

$$\psi_i(vp_t) = \tilde{\psi}_i(p_t) = 0, \quad i = 1, \dots, l.$$

As  $F(t) := |\psi(vp_t)| - |\psi(v(1-p_t))|$  is continuous and we have  $F(0) = -|\psi(v)|$ ,  $F(1) = |\psi(v)|$ , there exists  $t \in [0, 1]$  with  $|\psi(vp_t)| = |\psi(v(1-p_t))|$ . Hence  $p := p_t$  satisfies the requirement of the claim.

For this  $p$ , there exists  $\theta \in [0, 2\pi)$  with

$$\psi(vp) + e^{i\theta} \psi(v(1-p)) = 0.$$

Define

$$\bar{v} := vp + e^{i\theta} v(1-p) \in M.$$

Then we have

$$\bar{v}^* \bar{v} = v^* v, \quad \bar{v} \bar{v}^* = v v^*,$$

and

$$\psi_i(\bar{v}) = \psi(\bar{v}) = 0, \quad i = 1, \dots, l.$$

□

We now start constructing appropriate matrix units by induction both on the number of normal states and the size of matrix units.

**Lemma 2.6** *Let  $M$  be a factor of type III with separable predual, and  $\varphi_1, \dots, \varphi_k$  normal states on  $M$ . Let  $m \in \mathbb{N}$  and  $\rho$  a faithful state on  $M_m(\mathbb{C})$  with density matrix  $D_\rho = \sum_{i=1}^m \lambda_i f_{ii}$ , where  $\{f_{ij}\}_{i,j=1}^d$  is a standard system of matrix units in  $M_m(\mathbb{C})$ . Suppose that there exist mutually orthogonal nonzero projections  $e_{11}, \dots, e_{mm}$  in  $M$  with  $\sum_{i=1}^m e_{ii} = 1$  and for some  $1 \leq l < m$ , there exists a set of partial isometries  $\{u_{i1}\}_{i=1, \dots, l}$  in  $M$  satisfying the following identities.*

$$\begin{aligned} \varphi_n(e_{ii}) &= \lambda_i, \quad i = 1, \dots, m, \quad n = 1, \dots, k, \\ \varphi_n(u_{i1}u_{j1}^*) &= \delta_{ij}\lambda_i, \quad i, j = 1, \dots, l, \quad n = 1, \dots, k, \\ u_{i1}^*u_{i1} &= e_{11}, \quad u_{i1}u_{i1}^* = e_{ii}, \quad i = 1, \dots, l. \end{aligned}$$

*Then there exists a partial isometry  $u_{l+1,1}$  in  $M$  such that the above equations hold for  $i, j = 1, \dots, l+1$  and  $n = 1, \dots, k$ .*

**Proof.** Consider the following statement for  $h = 1, \dots, k$  and  $r = 1, \dots, l$ .

(B <sub>$h,r$</sub> ): There exists a partial isometry  $v$  in  $M$  such that  $v^*v = e_{11}$ ,  $vv^* = e_{l+1,l+1}$ , and we have  $\varphi(v) = 0$  for all  $\varphi$  in

$$\{\varphi_n(\cdot u_{j1}^*) \mid j = 1, \dots, r-1, \quad n = 1, \dots, k\} \cup \{\varphi_n(\cdot u_{r1}^*) \mid n = 1, \dots, h\}.$$

What we have to prove is (B <sub>$k,l$</sub> ). We prove this by induction first on  $h$  and next on  $r$ .

First, as  $M$  is a factor of type III, nonzero projections  $e_{11}$  and  $e_{l+1,l+1}$  are equivalent, i.e., there exists a partial isometry  $v$  in  $M$  such that  $e_{11} = v^*v$ ,  $e_{l+1,l+1} = vv^*$ . Applying Lemma 2.5 with  $l = 0$  to  $v$  and  $\varphi_1(\cdot u_{11}^*)$ , there exists a partial isometry  $\bar{v}$  satisfying  $\varphi_1(\bar{v}u_{11}^*) = 0$ ,  $\bar{v}^*\bar{v} = v^*v = e_{11}$  and  $\bar{v}\bar{v}^* = vv^* = e_{l+1,l+1}$ . Hence (B <sub>$1,1$</sub> ) is true.

Assume that (B <sub>$h,r$</sub> ) is true for  $h < k$ ,  $r \leq l$  or  $h = k$ ,  $r < l$  and let  $v$  be the partial isometry in  $M$  given by (B <sub>$h,r$</sub> ). Let  $\psi := \varphi_{h+1}(\cdot u_{r1}^*)$  if  $h < k$ , and  $\psi := \varphi_1(\cdot u_{r+1,1}^*)$  if  $h = k$ . Applying Lemma 2.5 to this  $\psi$ ,  $v$  and  $\sigma$ -weakly continuous linear functionals

$$\{\varphi_n(\cdot u_{j1}^*) \mid j = 1, \dots, r-1, \quad n = 1, \dots, k\} \cup \{\varphi_n(\cdot u_{r1}^*) \mid n = 1, \dots, h\},$$

we obtain a partial isometry  $\bar{v}$  satisfying  $\bar{v}^*\bar{v} = v^*v = e_{11}$ ,  $\bar{v}\bar{v}^* = vv^* = e_{l+1,l+1}$ ,  $\psi(\bar{v}) = 0$  and  $\varphi(\bar{v}) = 0$  for all  $\varphi$  in the above set.

Hence (B <sub>$h+1,r$</sub> ) holds if  $h < k$ , and (B <sub>$k,r+1$</sub> ) holds if  $h = k$  and  $r < l$ . We thus have (B <sub>$k,l$</sub> ) as desired.  $\square$

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** First we consider the case  $A = M_m(\mathbb{C})$ . We choose a system of matrix units  $\{v_{ij}\}_{i,j=1,\dots,m}$  of  $A = M_m(\mathbb{C})$  which diagonalizes the density matrix  $D_\rho$  of  $\rho$ , i.e.,  $D_\rho = \sum_{i=1}^m \lambda_i v_{ii}$ . As  $\rho$  is faithful, we have  $\lambda_i > 0$  for all  $i$ . We claim that there exist a system of matrix units  $\{e_{ij}\}_{i,j=1,\dots,m}$  in  $M$  satisfying

$$\varphi_n(e_{ij}) = \delta_{ij} \lambda_i, \quad n = 1, \dots, k, \quad i, j = 1, \dots, m. \quad (1)$$

To see this, first, from Proposition 2.3 (with  $l = 0$ ), we have mutually orthogonal projections  $e_{11}, \dots, e_{mm}$  in  $M$  satisfying  $\sum_{i=1}^m e_{ii} = 1$  and

$$\varphi_n(e_{ii}) = \lambda_i, \quad n = 1, \dots, k, \quad i = 1, \dots, m.$$

As  $\lambda_i > 0$ , each  $e_{ii}$  is a nonzero projection. We consider the following statement for  $r = 1, \dots, m$ .

(A<sub>r</sub>): There exists partial isometries  $\{u_{i1}\}_{i=1}^r$  satisfying  $u_{i1}^* u_{i1} = e_{11}$ ,  $u_{i1} u_{i1}^* = e_{ii}$  and  $\varphi_n(u_{i1} u_{j1}^*) = \delta_{ij} \lambda_i$ , for all  $n = 1, \dots, k$  and  $i, j = 1, \dots, r$ .

The statement (A<sub>1</sub>) is trivial, as we can set  $u_{11} := e_{11}$ . The fact that (A<sub>r</sub>) for  $r < m$  implies (A<sub>r+1</sub>) is Lemma 2.6. Hence we obtain (A<sub>m</sub>).

Set  $e_{ij} := u_{i1} u_{j1}^*$ . Then the system  $\{e_{ij}\}$  give matrix units satisfying (1).

Define

$$\pi : M_m(\mathbb{C}) \rightarrow M, \quad \pi(v_{ij}) = e_{ij}.$$

Then  $\pi$  gives a unital homomorphism satisfying the desired condition.

For the general case  $A \simeq \oplus_{k=1}^b M_{n_k}(\mathbb{C})$ , let  $m = \sum_{k=1}^b n_k$ . Let  $\hat{\rho}$  be a faithful extension of  $\rho$  to  $M_m(\mathbb{C})$ . Applying the above result to  $M_m(\mathbb{C})$  and  $\hat{\rho}$ , there exists a unital homomorphism  $\hat{\pi} : M_m(\mathbb{C}) \rightarrow M$  such that

$$\varphi_n \circ \hat{\pi} = \hat{\rho}, \quad n = 1, \dots, k.$$

The restriction  $\pi := \hat{\pi}|_A$  gives a unital homomorphism from  $A$  to  $M$  satisfying  $\varphi_n \circ \pi = \rho$ , for  $n = 1, \dots, k$ .  $\square$

Next we prove Corollary 2.2.

**Proof of Corollary 2.2.** Let  $p$  be the unit of  $A$ . Considering  $A \oplus \mathbb{C}(1 - p)$  instead of  $A$ , we may assume that  $A$  contains the unit of  $M$  from the beginning.

First we consider the case  $A \simeq M_m(\mathbb{C})$ ,  $m \in \mathbb{N}$ . Let  $\{f_{ij}\}_{i,j=1,\dots,m}$ ,  $\{v_{ij}\}_{i,j=1,\dots,m}$  be systems of matrix units of  $A$  and  $M_m(\mathbb{C})$ , respectively. Let  $\gamma : M_m(\mathbb{C}) \rightarrow A$  be an isomorphism given by  $\gamma(v_{ij}) = f_{ij}$ .

Then  $\rho := \omega \circ \gamma$  is a faithful state on  $M_m(\mathbb{C})$ . From Theorem 2.1, there exists a unital homomorphism  $\pi : M_m(\mathbb{C}) \rightarrow M$  such that  $\varphi_n \circ \pi = \rho$ ,  $n = 1, \dots, k$ . The algebras  $A$  and  $\pi(M_m(\mathbb{C}))$  are subalgebras of  $M$  isomorphic to  $M_m(\mathbb{C})$  with complete sets of matrix units  $\{f_{ij}\}$  and  $\{\pi(v_{ij})\}$ . As in [HM, Lemma 2.1], if  $v \in M$  is a partial isometry with  $v^* v = \pi(v_{11})$  and  $vv^* = f_{11}$ , then

$u := \sum_{i=1}^m \pi(v_{i1})v^* f_{1i}$  is a unitary in  $M$  satisfying  $u f_{ij} u^* = \pi(v_{ij})$ . Hence we have

$$\varphi_n \circ \text{Ad } u(f_{ij}) = \varphi_n(\pi(v_{ij})) = \rho(v_{ij}) = \omega \circ \gamma(v_{ij}) = \omega(f_{ij}),$$

i.e.,  $\varphi_n \circ \text{Ad } u|_A = \omega|_A$  for  $n = 1, \dots, k$ .

For the general case  $A \simeq \oplus_{l=1}^b M_{n_l}(\mathbb{C})$ , let  $\{f_{ij}^{(l)}\}_{ij=1, \dots, n_l}$  be a system of matrix units of  $M_{n_l}(\mathbb{C})$  for each  $l = 1, \dots, b$ . As  $M$  is of type III, for all  $l = 1, \dots, b$ , the nonzero projections  $f_{11}^{(l)}$  and  $f_{11}^{(l)}$  are mutually equivalent. Hence, there exist partial isometries  $v^{(l)} \in M$  such that  $v^{(l)*} v^{(l)} = f_{11}^{(l)}$  and  $v^{(l)} v^{(l)*} = f_{11}^{(1)}$ . Set  $w_{(k,i)(l,j)} := f_{i1}^{(k)} v^{(k)*} v^{(l)} f_{1j}^{(l)}$ , for  $k, l = 1, \dots, b$ ,  $i = 1, \dots, n_k$ , and  $j = 1, \dots, n_l$ . Then we have

$$\begin{aligned} w_{(k,i)(l,j)}^* &= f_{j1}^{(l)} v^{(l)*} v^{(k)} f_{1i}^{(k)} = w_{(l,j)(k,i)}, \\ w_{(k,i)(l,j)} w_{(l',j')(k',i')} &= f_{i1}^{(k)} v^{(k)*} v^{(l)} f_{1j}^{(l)} f_{j'1}^{(l')} v^{(l')*} v^{(k')} f_{1i'}^{(k')} \\ &= \delta_{ll'} \delta_{jj'} f_{i1}^{(k)} v^{(k)*} v^{(l)} f_{11}^{(l)} v^{(l)*} v^{(k')} f_{1i'}^{(k')} \\ &= \delta_{ll'} \delta_{jj'} f_{i1}^{(k)} v^{(k)*} v^{(l)} v^{(l)*} v^{(k')} f_{1i'}^{(k')} \\ &= \delta_{ll'} \delta_{jj'} w_{(k,i)(k',i')}, \\ \sum_{(k,i)} w_{(k,i)(k,i)} &= \sum_{i,k} f_{i1}^{(k)} v^{(k)*} v^{(k)} f_{1i}^{(k)} = \sum_{(k,i)} f_{ii}^{(k)} = 1. \end{aligned}$$

Hence  $\{w_{(k,i)(l,j)}\}_{(k,i)(l,j)}$  give a system of matrix units of a  $C^*$ -subalgebra  $B$  of  $M$  isomorphic to  $M_m$ , for  $m := \sum_{k=1}^b n_k$ . As  $w_{(ki)(kj)} = f_{i1}^{(k)} f_{1j}^{(k)} = f_{ij}^{(k)}$ ,  $\{w_{(k,i)(l,j)}\}$  is an extension of  $\{f_{ij}^{(k)}\}$  and  $A$  is a subalgebra of  $B$ . We apply the above argument to  $B \simeq M_m(\mathbb{C})$  and obtain a unitary  $u$  in  $M$  such that  $\varphi_i \circ \text{Ad } u|_B = \omega|_B$ . In particular, we obtain  $\varphi_i \circ \text{Ad } u|_A = \omega|_A$  for  $i = 1, \dots, k$ .  $\square$

The above theorem can be extended to the compact operators as follows.

**Theorem 2.7** *Let  $K(\mathcal{H})$  denote the set of all the compact operators on a separable Hilbert space  $\mathcal{H}$ . Let  $\rho$  be a faithful state on  $K(\mathcal{H})$ . Let  $M$  be a factor of type III with separable predual,  $\varphi_1, \varphi_2, \dots, \varphi_k$  normal states on  $M$ . Then there exists a homomorphism  $\pi$  of  $K(\mathcal{H})$  into  $M$  such that*

$$\varphi_n \circ \pi = \rho, \quad n = 1, \dots, k.$$

**Proof.** We may assume that  $\mathcal{H}$  is infinite dimensional, and  $\varphi_1$  is faithful, e.g. by adding a faithful state to the set of  $\varphi_i$ 's.

Let  $\{v_{ij}\}$  be a system of matrix units of  $K(\mathcal{H})$  diagonalizing the density matrix  $D_\rho$  of  $\rho$ , i.e.,  $D_\rho = \sum_{i=1}^\infty \lambda_i v_{ii}$ . As  $\rho$  is faithful, we have  $\lambda_i > 0$  for all  $i$ .

We claim that there exists a system of matrix units  $\{e_{ij}\}_{i,j \in \mathbb{N}}$  in  $M$  satisfying

$$\varphi_n(e_{ij}) = \delta_{ij} \lambda_i, \quad n = 1, \dots, k, \quad i, j = 1, \dots \quad (2)$$

To see this, first, from Proposition 2.3, we have mutually orthogonal projections  $e_{11}, e_{22}, \dots$  in  $M$  such that

$$\varphi_n(e_{ii}) = \lambda_i, \quad n = 1, \dots, k, \quad i = 1, \dots, .$$

From the construction, we see  $\sum_{i=1}^{\infty} e_{ii} = 1$ .

We consider the statements  $(A_r)$  in the proof of Theorem 2.1 for  $r = 1, 2, \dots$ . As in the proof of the latter Theorem,  $(A_m)$  is true for all  $m \in \mathbb{N}$ . Set  $e_{ij} := u_{i1}u_{j1}^*$  for  $i, j = 1, 2, \dots$ . Then  $\{e_{ij}\}$  is a system of matrix units satisfying (2). There exists a homomorphism  $\pi : K(\mathcal{H}) \rightarrow M$  with  $\pi(v_{ij}) = e_{ij}$ . This  $\pi$  satisfies  $\varphi_n \circ \pi = \rho$  for  $n = 1, \dots, k$ .  $\square$

### 3 Factors of type $II_1$

The analogue of Theorem 2.1 for semifinite factors is false. For example, if  $M$  is of type  $II_1$  with trace  $\tau$  and  $\rho$  is not a trace on  $A$ , then the conclusion of Theorem 2.1 for  $\varphi_1 = \tau$  is clearly false. However, if we restrict the choice of  $\omega$  in Corollary 2.2, we obtain a positive result.

**Theorem 3.1** *Let  $\varphi_1, \dots, \varphi_k$  be normal states on a factor  $M$  of type  $II_1$  with the unique trace  $\tau$ . Let  $A$  be a  $C^*$ -subalgebra of  $M$  isomorphic to  $M_m(\mathbb{C})$  with  $1 \in A$ . Then there exists a unitary operator  $u \in M$  satisfying  $\varphi_i \circ \text{Ad } u|_A = \tau|_A$  for  $i = 1, \dots, k$ .*

We first prove a lemma.

**Lemma 3.2** *Let  $M$  be a factor of type  $II_1$ , and  $\varphi_1, \dots, \varphi_k$  normal states on  $M$ , where  $\varphi_1 = \tau$  is the unique trace. Suppose that there exist mutually orthogonal nonzero projections  $e_{11}, \dots, e_{mm}$  in  $M$  with  $\sum_{i=1}^m e_{ii} = 1$  and for some  $1 \leq l < m$ , there exists a set of partial isometries  $\{u_{i1}\}_{i=1, \dots, l}$  in  $M$  satisfying*

$$\begin{aligned} \varphi_n(e_{ii}) &= \frac{1}{m}, \quad i = 1, \dots, m, \quad n = 1, \dots, k, \\ \varphi_n(u_{i1}u_{j1}^*) &= \delta_{ij} \frac{1}{m}, \quad i, j = 1, \dots, l, \quad n = 1, \dots, k, \\ u_{i1}^*u_{i1} &= e_{11}, \quad u_{i1}u_{i1}^* = e_{ii}, \quad i = 1, \dots, l. \end{aligned}$$

*Then there exists a partial isometry  $u_{l+1,1}$  in  $M$  such that the above equations hold for  $i, j = 1, \dots, l+1$  and  $n = 1, \dots, k$ .*

**Proof.** Consider the following statement for  $h = 1, \dots, k$  and  $r = 1, \dots, l$ .

$(B_{h,r})$ : There exists a partial isometry  $v$  in  $M$  satisfying  $v^*v = e_{11}$ ,  $vv^* = e_{l+1, l+1}$  and  $\varphi(v) = 0$  for all  $\varphi$  in

$$\{\varphi_n(\cdot u_{j1}^*) \mid j = 1, \dots, r-1, \quad n = 1, \dots, k\} \cup \{\varphi_n(\cdot u_{r1}^*) \mid n = 1, \dots, h\}.$$

First, as  $M$  is of type  $\text{II}_1$  and  $\tau(e_{11}) = \tau(e_{l+1,l+1}) = 1/m$ ,  $e_{11}$  and  $e_{l+1,l+1}$  are equivalent, i.e., there exists a partial isometry  $v$  in  $M$  with  $e_{11} = v^*v$ ,  $e_{l+1,l+1} = vv^*$ . Applying Lemma 2.5 with  $l = 0$  to  $v$  and  $\varphi_1(\cdot u_{11}^*)$ , there exists a partial isometry  $\bar{v}$  with  $\varphi_1(\bar{v}u_{11}^*) = 0$  and  $\bar{v}^*\bar{v} = v^*v = e_{11}$ ,  $\bar{v}\bar{v}^* = vv^* = e_{l+1,l+1}$ . Hence  $(B_{1,1})$  is true.

Assume that  $(B_{h,r})$  is true for  $h < k$ ,  $r \leq l$  or  $h = k$ ,  $r < l$  and let  $v$  be the partial isometry in  $M$  given by  $(B_{h,r})$ . Let  $\psi := \varphi_{h+1}(\cdot u_{r1}^*)$  if  $h < k$ , and  $\psi := \varphi_1(\cdot u_{r+1,1}^*)$  if  $h = k$ . Applying Lemma 2.5 to this  $\psi$ ,  $v$  and the  $\sigma$ -weakly continuous linear functionals

$$\{\varphi_n(\cdot u_{j1}^*) \mid j = 1, \dots, r-1, n = 1, \dots, k\} \cup \{\varphi_n(\cdot u_{r1}^*) \mid n = 1, \dots, h\},$$

we obtain a partial isometry  $\bar{v}$  satisfying  $\bar{v}^*\bar{v} = v^*v = e_{11}$ ,  $\bar{v}\bar{v}^* = vv^* = e_{l+1,l+1}$ ,  $\psi(\bar{v}) = 0$  and  $\varphi(\bar{v}) = 0$  for all  $\varphi$  in the above set.

Hence  $(B_{h+1,r})$  holds if  $h < k$ , and  $(B_{1,r+1})$  holds if  $h = k$  and  $r < l$ . Hence  $(B_{k,l})$ , which gives the claim of the Lemma, holds.  $\square$

Now we give a proof of Theorem 3.1.

**Proof of Theorem 3.1.** We may assume that  $\varphi_1 = \tau$  is the unique trace on  $M$ .

We claim that there exists a system of matrix units  $\{e_{ij}\}_{i,j=1,\dots,m}$  in  $M$  such that

$$\varphi_n(e_{ij}) = \delta_{ij} \frac{1}{m}, \quad n = 1, \dots, k, \quad i, j = 1, \dots, m. \quad (3)$$

To see this, first, from Proposition 2.3, we have mutually orthogonal projections  $e_{11}, \dots, e_{mm}$  in  $M$  such that  $\sum_{i=1}^m e_{ii} = 1$  and

$$\varphi_n(e_{ii}) = \frac{1}{m}, \quad n = 1, \dots, k, \quad i = 1, \dots, m.$$

We consider the following statement for  $r = 1, \dots, m$ .

$(A_r)$ : There exist partial isometries  $\{u_{i1}\}_{i=1}^r$  satisfying  $u_{i1}^*u_{i1} = e_{11}$ ,  $u_{i1}u_{i1}^* = e_{ii}$  and  $\varphi_n(u_{i1}u_{j1}^*) = \delta_{ij}/m$  for all  $n = 1, \dots, k$  and  $i, j = 1, \dots, r$ .

The statement  $(A_1)$  is trivial, as we can set  $u_{11} := e_{11}$ . Lemma 3.2 gives that  $(A_r)$  for  $r < m$  implies  $(A_{r+1})$ . Hence we obtain  $(A_m)$ .

Set  $e_{ij} := u_{i1}u_{j1}^*$ . Then  $\{e_{ij}\}$  is a system of matrix units satisfying (3).

We denote by  $B$  the  $C^*$ -subalgebra of  $M$  isomorphic to  $M_m(\mathbb{C})$  generated by  $\{e_{ij}\}$ . Let  $\{f_{ij}\}_{i,j=1,\dots,m}$  be a system of matrix units for  $A$ . As  $f_{ii} \sim f_{jj}$  for  $i, j = 1, \dots, m$  and  $1 = \sum_{i=1}^m f_{ii}$ , we have  $\tau(f_{11}) = \tau(f_{ii}) = 1/m$ . On the other hand, we have  $\tau(e_{ii}) = 1/m$  for  $i = 1, \dots, m$ . As  $M$  is a factor of type  $\text{II}_1$ , this means  $f_{ii} \sim e_{jj}$  for all  $i, j = 1, \dots, m$ . Furthermore, we have  $1 = \sum_{i=1}^m e_{ii} = \sum_{i=1}^m f_{ii}$ . As in [HM, Lemma 2.1], if  $v \in M$  is a partial isometry with  $v^*v = f_{11}$  and  $vv^* = e_{11}$ , then  $u := \sum_{i=1}^n e_{i1}vf_{1i}$  is a unitary operator in  $M$  satisfying  $u^*f_{ij}u = e_{ij}$ . Hence we have

$$\varphi_n \circ \text{Ad } u(f_{ij}) = \varphi_n(e_{ij}) = \delta_{ij} \frac{1}{m} = \tau(f_{ij}).$$

i.e.,  $\varphi_n \circ \text{Ad } u|_A = \tau|_A$  for  $n = 1, \dots, k$ . □

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